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Smoothing for Multipoint Boundary Value Models*

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Abstract

We show how to solve the linear least-squares smoothing problem for multipoint boundary value models. The complementary model is derived and is used to determine the Hamiltonian equations for the smoothed state estimate and its error covariance. Stable algorithms are obtained using an invariant imbedding/multiple shooting procedure.

Key words: boundary value models, acausal models, smoothing

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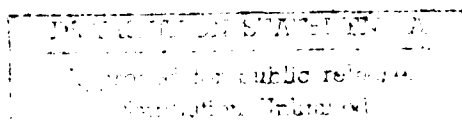
1. Introduction

Many physical systems can be modeled by linear partial differential equations with various types of boundary conditions. When such systems are subject to random excitations, it is often necessary (for reliability analysis or control, for example) to find the smoothed estimate of the state vector based on noisy observations. In certain cases, such multidimensional models can be Fourier transformed into sets of uncoupled ordinary differential equation models (see, for example, [RW1], [RW2], [LAW]). For ordinary differential equations with two-point boundary conditions, the smoothing problem has been studied in [AWL], [RW1], [RW2], [BW], [NAWL], [W], [LAW], [LFK].

We intend in this paper to investigate the case of multipoint boundary conditions. One context in which multipoint boundary conditions arise is that of flexible structures, like bridges and moorings, which consist of components coupled end to end (see, for example, [CDKP]). Using the concept of the complementary model, introduced in [WD], along with invariant imbedding and multiple shooting, we will derive efficient, stable algorithms for the smoothed state estimate and its error covariance. As we shall see, the multipoint case has certain characteristics which do not appear in the two-point case.

The multipoint boundary value model of interest here is given by

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$$\dot{z}(x) = Az(x) + Bu(x), \quad x \in [x_0, x_{N-1}] \qquad (1)$$

$$\sum_{i=0}^{N-1} V_i z(x_i) = r, \quad x_0 < x_1 < \ldots < x_{N-1} \qquad (2)$$

where x is typically a spatial variable and u is a zero-mean, unit intensity white noise with m components, r is a zero-mean random n-vector with

$$Err' = Q$$

z is a continuous n-component state vector, and A, B, V_0, $V_1, \ldots, V_{N-1}$ are constant matrices of appropriate size. We assume that u and r are uncorrelated. We have continuous measurements given by

$$y(x) = Cz(x) + v(x), \quad x \in [x_0, x_{N-1}] \qquad (3)$$

where C is a pxn matrix and v is a zero-mean, unit intensity white noise which is uncorrelated with u and r. Given the measurements y, our objective is to find the smoothed estimate (in the linear least-squares sense) of the state z, and its associated error covariance.

As a simple example of a multipoint boundary value model, consider a static Euler-Bernoulli beam with flexural rigidity β and a continuously distributed load with density u. In state space form, the beam model is given by eqn. (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\beta \end{bmatrix}$$

where the first component of the state vector is the deflection of the beam at point x. Now suppose there are boundary conditions consisting of pins at x_0 and x_2 and a clamp at x_1. Then these can be represented by eqn. (2) with

$$V_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and with Q quantifying the looseness, or uncertainty, of the pins and clamp.

2. The Complementary Model

The Hamiltonian equations for the smoothed estimate are most readily derived using the method of complementary models [WD], [AWL], [RW2], [AK]. If Y is the closed linear span of the components of y, then a complementary model is, in the case at hand, a multipoint boundary value model which generates complementary variables, denoted y_c and θ, whose components span the orthogonal complement of Y. In order to derive a complementary model, we must first solve for z in terms of r and u.

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Integrating eqn. (1) gives

$$z(x) = \Phi(x, x_0)z(x_0) + \int_{x_0}^{x} \Phi(x, \xi)Bu(\xi)d\xi \tag{4}$$

where Φ is the state transition matrix. Substituting into eqn. (2) yields

$$r = \sum_{j=0}^{N-1} V_j \Phi(x_j, x_0)z(x_0) + \sum_{j=0}^{N-1} V_j \int_{x_0}^{x_j} \Phi(x_j, \xi)Bu(\xi)d\xi \tag{5}$$

The boundary value problem will be well-posed (that is, u and r will give rise to a unique z) if and only if the matrix

$$\sum_{j=0}^{N-1} V_j \Phi(x_j, x_0)$$

is nonsingular. By multiplying eqn. (2) on the left by the inverse of this matrix, we can always ensure that

$$\sum_{j=0}^{N-1} V_j \Phi(x_j, x_0) = I \tag{6}$$

In what follows it will be assumed that eqn. (6) holds.

Solving eqn. (5) for $z(x_0)$ and substituting into eqn. (4), we get

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$$z(x) = \Phi(x,x_0)r + \int_{x_0}^{x_{N-1}} G(x,\xi)Bu(\xi)d\xi \tag{7}$$

where the multipoint Green's function $G(x,\xi)$ is given by

$$G(x,\xi) = \begin{cases} \Phi(x,x_0)\sum_{k=0}^{J}V_k\Phi(x_k,\xi), & x > \xi \\ -\Phi(x,x_0)\sum_{k=J+1}^{N-1}V_k\Phi(x_k,\xi), & x < \xi \end{cases} \quad x_J < \xi < x_{J+1} \tag{8}$$

for j=0,1,....,N-2. By analogy with previous work [WD], we can deduce from the form of eqns. (3) and (7) that complementary variables are given by

$$y_c(x) = u(x) - B'\int_{x_0}^{x_{N-1}} G'(\xi,x)C'v(\xi)d\xi \tag{9}$$

$$D\theta = r - Q\int_{x_0}^{x_{N-1}} \Phi'(\xi,x_0)C'v(\xi)d\xi \tag{10}$$

where D is any full-rank matrix satisfying

$$DD' = Q$$

Defining a complementary state by

$$z_c(x) = \int_{x_0}^{x_{N-1}} G'(\xi,x)C'v(\xi)d\xi \tag{11}$$

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we see that

$$y_c(x) = u(x) - B'z_c(x) \tag{12}$$

Note that in contrast to the two-point case in which it is continuous, the complementary state here has jumps at the points $x=x_1,x_2,....x_{N-2}$.

Just as with eqn. (9), we would like to express eqn. (10) in terms of z_c. Defining

$$\delta_j = z_c(x_j-) - z_c(x_j+), \quad j = 1,2,....,N-2 \tag{13}$$

and using eqns. (8) and (11), we find that

$$z_c(x_{N-1}) = -V'_{N-1}\int_{x_0}^{x_{N-1}} \Phi'(\xi,x_0)C'v(\xi)d\xi \tag{14}$$

$$\delta_j = -V'_j\int_{x_0}^{x_{N-1}} \Phi'(\xi,x_0)C'v(\xi)d\xi \tag{15}$$

$$z_c(x_0) = V'_0\int_{x_0}^{x_{N-1}} \Phi'(\xi,x_0)C'v(\xi)d\xi \tag{16}$$

Using the above along with eqn. (6), we can rewrite eqn. (10) as

$$D\theta = r - Q[z_c(x_0) - \sum_{j=1}^{N-2}\Phi'(x_j,x_0)\delta_j - \Phi'(x_{N-1},x_0)z_c(x_{N-1})] \tag{17}$$

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Next, we want to express z_c as the state vector of a multipoint boundary value model like eqns. (1)-(2). Differentiating eqn. (11) gives

$$\dot{z}_c(x) = -A'z_c(x) - C'v(x), \quad x \in [x_0, x_{N-1}] \setminus \{x_j\}_{j=1}^{N-2} \tag{18}$$

The boundary conditions that go with eqn. (18) must yield a unique z_c which matches eqn. (11). Since eqn. (18) is valid in each of N-1 subintervals, we need a total of (N-1)n independent boundary conditions for a unique solution. Therefore the boundary conditions for the complementary state must have the form

$$K_0 z_c(x_0) + \sum_{j=1}^{N-2} K_j \delta_j + K_{N-1} z_c(x_{N-1}) = 0 \tag{19}$$

for some suitable set of matrices $\{K_j\}$ of size (N-1)n×n. In fact, if

$$
\begin{aligned}
K_0 &= \begin{bmatrix} V_1 & V_2 & \cdots & V_{N-1} \end{bmatrix}' \\
K_1 &= \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}' - K_0 \Phi'(x_1, x_0) \\
&\vdots \\
K_{N-1} &= \begin{bmatrix} 0 & \cdots & 0 & I \end{bmatrix}' - K_0 \Phi'(x_{N-1}, x_0)
\end{aligned}
\tag{20}
$$

it can be verified (see the appendix) that the boundary value problem of

eqns. (18)-(19) is well-posed and that its solution matches eqn. (11). Eqns. (12), (17)-(20) constitute a complementary model for eqns. (1)-(3).

3. The Hamiltonian Equations

The value of a complementary model now becomes clear because by cross-substituting between the original model and the complementary model, we can eliminate u, r, and v, and then project onto the space spanned by the observations to obtain the Hamiltonian equations

$$\dot{\hat{z}}(x) = A\hat{z}(x) + BB'\hat{z}_c(x), \quad x \in [x_0, x_{N-1}] \tag{21}$$

$$\dot{\hat{z}}_c(x) = -A'\hat{z}_c(x) + C'C\hat{z}(x) - C'y(x), \quad x \in [x_0, x_{N-1}] \setminus \{x_j\}_{j=1}^{N-2} \tag{22}$$

with boundary conditions

$$\sum_{j=0}^{N-1} \begin{bmatrix} V_j \\ 0 \end{bmatrix} \hat{z}(x_j) + \begin{bmatrix} -Q \\ K_0 \end{bmatrix} \hat{z}_c(x_0) + \sum_{j=1}^{N-2} \begin{bmatrix} Q\Phi'(x_j, x_0) \\ K_j \end{bmatrix} \hat{\delta}_j + \begin{bmatrix} Q\Phi'(x_{N-1}, x_0) \\ K_{N-1} \end{bmatrix} \hat{z}_c(x_{N-1}) = 0 \tag{23}$$

where the "hat" denotes linear least-squares estimate. Note that since the complementary variables y_c and θ are uncorrelated with y, their estimates are zero.

By using invariant imbedding, we can transform eqns. (21)-(23) so that the resulting differential equations are partially decoupled and asymp-

totically stable. Let $N(x)$ be the solution to the following Riccati equation

$$\dot{N}(x) = -N(x)A - A'N(x) + N(x)BB'N(x) - C'C$$
$$N(x_{N-1}) = 0$$

(24)

Then the coordinate change

$$\rho(x) = N(x)\hat{z}(x) + \hat{z}_c(x)$$

transforms the above Hamiltonian equations into

$$\dot{\hat{z}}(x) = (A - BB'N(x))\hat{z}(x) + BB'\rho(x), \quad x \in [x_0, x_{N-1}]$$

(25)

$$\dot{\rho}(x) = -(A - BB'N(x))'\rho(x) - C'y(x), \quad x \in [x_0, x_{N-1}] \setminus \{x_j\}_{j=1}^{N-2}$$

(26)

with boundary conditions

$$\begin{bmatrix} V_0 + QN(x_0) & -Q \\ -K_0 N(x_0) & K_0 \end{bmatrix} \begin{bmatrix} \hat{z}(x_0) \\ \rho(x_0) \end{bmatrix} + \sum_{j=1}^{N-1} \begin{bmatrix} V_j & Q\Phi'(x_j, x_0) \\ 0 & K_j \end{bmatrix} \begin{bmatrix} \hat{z}(x_j) \\ \delta_{\rho_j} \end{bmatrix} = 0$$

(27)

where we have defined

$$\delta_{\rho_j} = \begin{cases} \rho(x_j-) - \rho(x_j+), & j = 1, 2, \dots, N-2 \\ \rho(x_{N-1}), & j = N-1 \end{cases}$$

If (A, B) is stabilizable and (A, C) is detectable, then eqns. (25) and (26) will

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be asymptotically stable in the directions of increasing and decreasing x, respectively. Note that the Riccati equation is identical to one that arises in smoothing for initial value models [WD], [AK]. Other invariant imbedding transformations are possible, for example using two Riccati equations, but the one used here is the most efficient in terms of the amount of computation required to find the smoothed estimate and its error covariance [W].

4. The Smoothed State Estimate

The transformed Hamiltonian equations can now be solved using a multiple shooting procedure. First solve the following initial value systems:

$$\dot{\rho}_0(x) = -(A - BB'N(x))'\rho_0(x) - C'y(x), \quad \rho_0(x_{N-1}) = 0 \tag{28}$$

$$\dot{\hat{z}}_0(x) = (A - BB'N(x))\hat{z}_0(x) + BB'\rho_0(x), \quad \hat{z}_0(x_0) = 0 \tag{29}$$

The solution of eqn. (29) can be interpreted as the smoothed estimate of $z(x)$ when $V_0 = I$, $V_j = 0$ for $j > 0$, and $Q = 0$; or in other words, when the original model is an initial value model with zero initial condition [WD].

Now it can be checked by differentiation that for $x_{k-1} \leq x \leq x_k$, $1 \leq k \leq N-1$,

$$\hat{z}(x) = \hat{z}_0(x) + \Psi(x, x_0)\hat{z}(x_0) + \sum_{j=1}^{k-1} \Psi(x, x_j)M(x_j)\delta_{\rho_j} + \sum_{j=k}^{N-1} M(x)\Psi'(x_j, x)\delta_{\rho_j} \tag{30}$$

and for $x_{k-1} < x < x_k$,

$$\rho(x) = \rho_0(x) + \sum_{j=k}^{N-1} \Psi'(x_j, x) \delta_{\rho_j} \qquad (31)$$

where Ψ is the state transition matrix of $(A - BB'N(x))$ and

$$\dot{M}(x) = (A - BB'N(x))M(x) + M(x)(A - BB'N(x))' + BB' \qquad (32)$$
$$M(x_0) = 0$$

We can determine the unknown quantities in eqn. (30) by obtaining $\hat{z}(x_k)$, $k=1,2,....,N-1$, and $\rho(x_0)$ from eqns. (30) and (31), respectively, and substituting into eqn. (27). This results in the following system:

$$[I + F_1 F_2] \begin{bmatrix} \hat{z}(x_0) \\ \delta_{\rho_1} \\ \vdots \\ \delta_{\rho_{N-1}} \end{bmatrix} = F_1 \begin{bmatrix} \rho_0(x_0) \\ \hat{z}_0(x_1) \\ \vdots \\ \hat{z}_0(x_{N-1}) \end{bmatrix} \qquad (33)$$

where

$$F_1 = \begin{bmatrix} Q & -K_0' \\ -K_0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} N(x_0) & -\Delta' \\ -\Delta & -L \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Psi(x_1, x_0) - \Phi(x_1, x_0) \\ \vdots \\ \Psi(x_{N-1}, x_0) - \Phi(x_{N-1}, x_0) \end{bmatrix}$$

$$L = \begin{bmatrix} M(x_1) & M(x_1)\Psi''(x_2,x_1) & \cdots & M(x_1)\Psi''(x_{N-1},x_1) \\ \Psi(x_2,x_1)M(x_1) & M(x_2) & \cdots & M(x_2)\Psi''(x_{N-1},x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi(x_{N-1},x_1)M(x_1) & \Psi(x_{N-1},x_2)M(x_2) & \cdots & M(x_{N-1}) \end{bmatrix}$$

The well-posedness of eqns. (1)-(2) together with the uniqueness of linear least-squares estimates guarantee the nonsingularity of $(I+F_1F_2)$. The components of the vector on the right side of eqn. (33) are obtained from eqns. (28)-(29). The solution of eqn. (33) is used in eqn. (30) to produce the smoothed estimate.

5. The Error Covariance

The smoothing error $\tilde{z}(x)$ is defined by

$$\tilde{z}(x) = z(x) - \hat{z}(x)$$

Eqns. (1)-(3) and (21)-(23) imply that the smoothing error is also characterized by a multipoint boundary value problem as follows:

$$\dot{\tilde{z}}(x) = A\tilde{z}(x) + BB'(-\hat{z}_c(x)) + Bu(x), \quad x \in [x_0,x_{N-1}]$$

$$-\dot{\hat{z}}_c(x) = -A'(-\hat{z}_c(x)) + C'C\tilde{z}(x) + C'v(x), \quad x \in [x_0,x_{N-1}] \setminus \{x_j\}_{j=1}^{N-2}$$

$$\sum_{j=0}^{N-1} \begin{bmatrix} V_j \\ 0 \end{bmatrix} \tilde{z}(x_j) + \begin{bmatrix} -Q \\ K_0 \end{bmatrix} (-\hat{z}_c(x_0)) + \sum_{j=1}^{N-2} \begin{bmatrix} Q\Phi'(x_j,x_0) \\ K_j \end{bmatrix} (-\hat{\delta}_j) + \begin{bmatrix} Q\Phi'(x_{N-1},x_0) \\ K_{N-1} \end{bmatrix} (-\hat{z}_c(x_{N-1})) = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

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A change of coordinates with

$$\xi(x) = N(x)\tilde{z}(x) - \hat{z}_c(x)$$

$$\delta_{\xi_j} = \begin{cases} \xi(x_j-) - \xi(x_j+), & j = 1,2,...,N-2 \\ \xi(x_{N-1}), & j = N-1 \end{cases}$$

transforms the above equations into

$$\dot{\tilde{z}}(x) = (A - BB'N(x))\tilde{z}(x) + BB'\xi(x) + Bu(x), \quad x \in [x_0, x_{N-1}] \tag{34}$$

$$\dot{\xi}(x) = -(A - BB'N(x))'\xi(x) + N(x)Bu(x) + C'v(x), \quad x \in [x_0, x_{N-1}] \setminus \{x_j\}_{j=1}^{N-2} \tag{35}$$

$$\begin{bmatrix} V_0 + QN(x_0) & -Q \\ -K_0 N(x_0) & K_0 \end{bmatrix} \begin{bmatrix} \tilde{z}(x_0) \\ \xi(x_0) \end{bmatrix} + \sum_{j=1}^{N-1} \begin{bmatrix} V_j & Q\Phi'(x_j, x_0) \\ 0 & K_j \end{bmatrix} \begin{bmatrix} \tilde{z}(x_j) \\ \delta_{\xi_j} \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \tag{36}$$

As before, if we let

$$\dot{\xi}_0(x) = -(A - BB'N(x))'\xi_0(x) + N(x)Bu(x) + C'v(x), \quad \xi_0(x_{N-1}) = 0 \tag{37}$$

$$\dot{\tilde{z}}_0(x) = (A - BB'N(x))\tilde{z}_0(x) + BB'\xi_0(x) + Bu(x), \quad \tilde{z}_0(x_0) = 0 \tag{38}$$

then for $x_{k-1} \leq x \leq x_k$, $1 \leq k \leq N-1$,

$$\tilde{z}(x) = \tilde{z}_0(x) + \Psi(x, x_0)\tilde{z}(x_0) + \sum_{j=1}^{k-1} \Psi(x, x_j)M(x_j)\delta_{\xi_j} + \sum_{j=k}^{N-1} M(x)\Psi'(x_j, x)\delta_{\xi_j} \tag{39}$$

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where

$$[I + F_1 F_2] \begin{bmatrix} \tilde{z}(x_0) \\ \delta_{\xi_1} \\ \vdots \\ \delta_{\xi_{N-1}} \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} + F_1 \begin{bmatrix} \xi_0(x_0) \\ \tilde{z}_0(x_1) \\ \vdots \\ \tilde{z}_0(x_{N-1}) \end{bmatrix} \tag{40}$$

From [WD] we know that

$$E[\xi_0(x_0)\xi_0'(x_0)] = N(x_0), \quad E[\tilde{z}_0(x)\xi_0'(x_0)] \equiv 0 \tag{41}$$

$$E[\tilde{z}_0(x)\tilde{z}_0'(s)] = \begin{cases} \Psi(x,s)M(s), & x \geq s \\ M(x)\Psi''(s,x), & x \leq s \end{cases} \tag{42}$$

Therefore

$$P(x) = E[\tilde{z}(x)\tilde{z}'(x)]$$

$$= M(x) + R(x)(I + F_1 F_2)^{-1} \left[\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} + F_1 \begin{bmatrix} N(x_0) & 0 \\ 0 & L \end{bmatrix} F_1 \right] (I + F_1 F_2)'^{-1} R'(x)$$

$$- R(x)(I + F_1 F_2)^{-1} \begin{bmatrix} K_0' S'(x) \\ 0 \end{bmatrix} - [S(x)K_0 \quad 0](I + F_1 F_2)'^{-1} R'(x) \tag{43}$$

where for $x_{k-1} \leq x \leq x_k$, $1 \leq k \leq N-1$,

$$S(x) = \begin{bmatrix} \Psi(x,x_1)M(x_1) & \cdots & \Psi(x,x_{k-1})M(x_{k-1}) & M(x)\Psi''(x_k,x) & \cdots & M(x)\Psi''(x_{N-1},x) \end{bmatrix}$$

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and

$$R(x) = \begin{bmatrix} \varPsi(x,x_0) & S(x) \end{bmatrix}$$

Appendix

From eqn. (18),

$$z_c(x_0) = \varPhi'(x_1,x_0)z_c(x_1-) + \int_{x_0}^{x_1} \varPhi'(\xi,x_0)C'v(\xi)d\xi$$

But

$$z_c(x_1-) = \delta_1 + z_c(x_1+)$$

and from eqn. (18)

$$z_c(x_1+) = \varPhi'(x_2,x_1)z_c(x_2-) + \int_{x_1}^{x_2} \varPhi'(\xi,x_1)C'v(\xi)d\xi$$

Therefore

$$z_c(x_0) = \varPhi'(x_1,x_0)\delta_1 + \varPhi'(x_2,x_0)z_c(x_2-) + \int_{x_0}^{x_2} \varPhi'(\xi,x_0)C'v(\xi)d\xi$$

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Continuing in the same fashion, we get

$$z_c(x_0) = \sum_{j=1}^{N-2} \Phi'(x_j,x_0)\delta_j + \Phi'(x_{N-1},x_0)z_c(x_{N-1}) + \int_{x_0}^{x_{N-1}} \Phi'(\xi,x_0)C'v(\xi)d\xi$$

Substituting into eqn. (19) gives

$$\sum_{j=1}^{N-2} [K_j + K_0\Phi'(x_j,x_0)]\delta_j + [K_{N-1} + K_0\Phi'(x_{N-1},x_0)]z_c(x_{N-1}) + K_0\int_{x_0}^{x_{N-1}} \Phi'(\xi,x_0)C'v(\xi)d\xi = 0$$

With the $\{K_j\}$ given by eqn. (20), $z_c(x_{N-1})$ and the $\{\delta_j\}$ can be obtained uniquely from the above equation, and in fact are given by eqns. (14)-(15). Integrating eqn. (18) with the values given in eqns. (14)-(15) will produce eqn. (11). Therefore the boundary value problem of eqns. (18)-(19) is well-posed and its solution is given by eqn. (11).

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WE SHOW HOW TO SOLVE THE LINEAR LEAST-SQUARES SMOOTHING PROBLEM FOR MULTIPOINT BOUNDARY VALUE MODELS. THE COMPLEMENTARY MODEL IS DERIVED AND IS USED TO DETERMINE THE HAMILTONIAN EQUATIONS FOR THE SMOOTHED STATE ESTIMATE AND ITS ERROR COVARIANCE. STABLE ALGORITHMS ARE OBTAINED USING AN INVARIANT IMBEDDING/MULTIPLE SHOOTING PROCEDURE.

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